Bogomolov Instability

X smooth projective surface, E rk 2 v.b. on X

E is Bogomolov unstable if $\exists$ $Z \subset X$ finite (possibly empty) and l.b. A, B on X s.t.

$$(*) \quad 0 \to A \to E \to B \otimes I_Z \to 0 \quad \text{is exact}$$

and $(A-B)^2 > 0$ and $(A-B)\cdot H > 0$ for all ample H.

"A is more positive than B"
(Recall: Z is vanishing of $A \to E$)

Note: Bogomolov instability is invariant under tensoring by a line bundle.

Bogomolov's Instability Theorem: Let E be a rank 2 vector bundle on X a sm. projective surface. If $c_1(E)^2 - 4c_2(E) > 0$ then E is Bogomolov unstable.

Note: This implies that $c_1(E)^2 - 4c_2(E)$ is invariant under twisting by a line bundle.

Exercise: Show this is the only deg 2 homogeneous polynomial that has this property.

Feb 10

If E sits in a SES like $(*)$, then what are chern classes?

Twist by A^*:

$$0 \to \mathcal{O} \to A^* \otimes E \to (A^* \otimes B) \otimes I_Z \to 0$$

Recall that this comes from Koszul complex

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] & A^* \otimes E \arrow[rr] \arrow[dr] & & \det(A^* \otimes E) \\
& & & \det(A^* \otimes E) \otimes I_Z \arrow[ur, hook] \arrow[dr] & \\
& & & & 0
\end{tikzcd}$$

So $\det(A^* \otimes E) \cong A^* \otimes B \Rightarrow (A^*)^{\otimes 2} \otimes \det(E) \cong A^* \otimes B$

$\Rightarrow \det(E) \cong A \otimes B$

Thus $c_1(E) = c_1(A) + c_1(B)$

Since $rk(E) = 2$, $c_2(A^* \otimes E) =$ length of vanishing of section $\mathcal{O} \to E$ $=$ length(Z)

By the splitting principle,

$$c_2(A^* \otimes E) \left(= c_2\left((A^* \otimes e_1) \oplus (A^* \otimes e_2)\right) = c_1(A^* \otimes e_1)\, c_1(A^* \otimes e_2)\right)$$
$$= c_1(A^*)^2 + c_1(E)\, c_1(A^*) + c_2(E)$$

$$\Rightarrow c_2(E) = \text{length}(Z) - c_1(A^*)^2 - \left(c_1(A) + c_1(B)\right) c_1(A^*)$$
$$= \text{length}(Z) - c_1(A)^2 + c_1(A)^2 + c_1(A)\, c_1(B)$$

$$= \text{length}(Z) + c_1(A)c_1(B)$$